# S. BROVERMAN STUDY GUIDE

# FOR

# SOA EXAM FM/CAS EXAM 2

2025 EDITION

Samuel Broverman, ASA, PHD

2brove@rogers.com
www.sambroverman.com

2brove@rogers.com
www.sambroverman.com

# S. BROVERMAN EXAM FM/2 STUDY GUIDE
**2025**

## TABLE OF CONTENTS

# INTRODUCTORY NOTE

This study guide is designed to help with the preparation for Exam FM of the Society of Actuaries. This study guide is based on the February 2025 syllabus of the SOA.

Among the references indicated by the SOA in the exam syllabus for Exam FM is the 8th edition of *Mathematics of Investment and Credit* by S. Broverman. Along with the SOA listed Text References, there is a SOA Study Note as part of the readings for the FM exam. The notation and ordering of material will be mostly consistent with those references. This study guide covers all of the mathematical content of the the Study Note titled "FM-24-17 Using Duration and Convexity to Approximate Change in Present Value". It is recommended that those preparing for the FM exam read the Study Note.

Many examples in this study guide are labeled "SOA", and some, "EA1". This indicates that the example is from a previous SOA or Enrolled Actuaries exam. The review part of this study guide is divided into 14 sections, each with a problem set, plus a Problem Set 15 which has exercises specifically for practice on the BA II PLUS calculator. This is followed by 8 practice exams. The exam is scheduled to be 2.5 hours long with 30 multiple choice questions. In addition, at the start of each section of notes is a reference to the appropriate sections in *Mathematics of Investment and Credit* in which the material is covered.

Because of the time constraint on the exam, a crucial aspect of exam taking is the ability to work quickly. I believe that working through many problems and examples is a good way to build up the speed at which you work. It can also be worthwhile to work through problems that have been done before, as this helps to reinforce familiarity, understanding and confidence. Working many problems will also help in being able to more quickly identify topic and question types.

I have attempted, wherever possible, to emphasize shortcuts and efficient and systematic ways of setting up solutions. There are also occasional comments on interpretation of the language used in some exam questions. While the focus of the study guide is on exam preparation, from time to time there will be comments on underlying theory in places that I feel those comments may provide useful insight into a topic.

This study guide contains 65 detailed examples throughout the sections of review notes, and over 215 exercises in the problem sets. There are 30 questions on each of the 8 practice exams. Some of the questions on the practice exams have been taken or adapted from past SOA exams. SOA exam questions are copyrighted material of the SOA. The author gratefully acknowledges the permission granted by the SOA to use those questions in this study guide.

Although many examples in the review notes and some of the exercises are from old SOA exams, some of the problem set exercises are more in depth than actual exam questions, but the practice exam questions have been created in an attempt to replicate the breadth of topic and level of depth and difficulty of actual exam questions. It is worthwhile to occasionally check the SOA websites to see if any new sample questions have been posted.

It has been my intention to make this study guide self-contained and comprehensive for Exam FM/CAS 2. While the ability to derive formulas used on the exam is usually not the focus of an exam question, it can be useful in enhancing the understanding of the material and may be helpful in memorizing formulas. There will be some references in the review notes to derivations, but you are encouraged to review the official reference material for more detail on formula derivations. In order for this study guide to be most effective, you should be reasonably familiar with differential and integral calculus.

Of the various calculators that are allowed for use on Exam FM, I think that the BA II PLUS is probably the best choice. It has several easily accessible memories. It is probably the most functional of all the calculators allowed. Throughout the notes you will find boxed examples labeled "Calculator Notes, BA II PLUS". I am the author of a couple of study notes available at the SOA website which cover applications of the BA II PLUS and BA-35 calculators in some detail (the BA-35 is no longer displayed at the Texas Instruments website, so it is possible that TI no longer produces it). These study notes are available at www.soa.org . The study notes provide examples of keystroke sequences for solving the more frequently encountered types of calculations for that calculator. Students are strongly urged to review the calculator functions in the calculator guidebooks.

The calculator guidebook should come with the calculator when it is purchased, or can be downloaded from the Texas Instruments website:
https://education.ti.com/en/guidebook/search/financial-calculators .

If you have any questions, comments, criticisms or compliments regarding this study guide, you may contact me at the address below. I apologize in advance for any errors, typographical or otherwise, that you might find, and it would be greatly appreciated if you would bring them to my attention. I will be maintaining a website for errata that can be accessed from www.sambroverman.com (at the Exam FM link).

It is my sincere hope that you find this study guide helpful and useful in your preparation for the exam. I wish you the best of luck on the exam.

Samuel A. Broverman October 2024
Department of Statistics,
University of Toronto
100 St. George Street
Toronto, Ontario CANADA M5S 3G3
E-mail: 2brove@rogers.com or sam.broverman@utoronto.ca
Internet: www.sambroverman.com

# NOTES, EXAMPLES

# AND PROBLEM SETS

# SOA EXAM FM NOTES



## Calculus Review

**Natural log and exponential functions:** $ln(x)$ (sometimes written $log(x)$) is the log to the base $e$ ($e \approx 2.7183$) ; $ln(e) = 1$ , $ln(1) = 0$ , $e^0 = 1$ , $ln(e^y) = y$ , $e^{ln(x)} = x$ , $ln(a^y) = y \cdot ln(a)$ , $ln(y \cdot z) = ln(y) + ln(z)$ , $ln(\frac{y}{z}) = ln(y) - ln(z)$ , $e^x e^w = e^{x+w}$ , $(e^x)^w = e^{x \cdot w}$

**Differentiation:** product rule $\frac{d}{dx}[g(x) \cdot h(x)] = g'(x)h(x) + g(x)h'(x)$ ;
quotient rule $\frac{d}{dx}[\frac{h(x)}{g(x)}] = \frac{h'(x)g(x) - g'(x)h(x)}{[g(x)]^2}$ ;
$\frac{d}{dx} ln[g(x)] = \frac{g'(x)}{g(x)}$ ; $\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)$ ; $\frac{d}{dx} a^x = a^x \cdot ln(a)$

**Integration:** $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$ ; $\int a^x \, dx = \frac{a^x}{ln(a)} + c$ ; $\int \frac{1}{a+bx} \, dx = \frac{1}{b} ln(a+bx) + c$ ;
$\frac{d}{dx} \int_a^x g(t) \, dt = g(x)$ ; $\frac{d}{dx} \int_x^b g(t) \, dt = -g(x)$ ;
if $k > 0$ then $\int_0^\infty e^{-kx} \, dx = \frac{1}{k}$ ;
if $k > 0$ and $n$ is an integer $\geq 0$ then $\int_0^\infty x^n e^{-kx} \, dx = \frac{n!}{k^{n+1}}$

**Some useful finite and infinite series:** The following series summations may arise on an exam question:

(i) **sum of the first $n$ positive integers**: $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

(ii) **finite geometric series:** $1 + r + r^2 + \cdots + r^k = \frac{1-r^{k+1}}{1-r}$ , $r + r^2 + \cdots + r^k = \frac{r-r^{k+1}}{1-r}$

(iii) **infinite geometric series:** if $|r| < 1$ then $1 + r + r^2 + \cdots = \frac{1}{1-r}$ , $r + r^2 + \cdots = \frac{r}{1-r}$

(iv) **increasing geometric series:** infinite series, $1 + 2r + 3r^2 + \cdots = \frac{1}{(1-r)^2}$ ,

(v) **some series that are less likely to arise:**

exponential series, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

natural log series, if $|x| < 1$ then $ln(1+x) = x - + \frac{x^2}{2} + \frac{x^3}{3} -$

## SECTION 1 - EFFECTIVE RATES OF INTEREST AND DISCOUNT

# SECTION 1 - EFFECTIVE RATES OF INTEREST AND DISCOUNT

**Sections 1.1-1.3 of "Mathematics of Investment and Credit"**

Of the various dictionary definitions of **interest** (in the context of a financial transaction) that can be found, a typical one is that interest is the charge for or cost of borrowing money over a period of time. The **sum or amount of interest charged** is related to the **rate of interest** that is applied to the amount borrowed over the time period. Interest rates are generally described in one of two equivalent ways:
(i) as a percentage, such as 10%, or (ii) as a decimal, such as .10 (equivalent to 10%).
A financial transaction can take place over any period of time, and interest rates can be quoted for any time period, but the conventional way in which an interest rate is quoted is as an annual rate.

In the case of a simple loan or investment transaction in which an amount is borrowed (or invested) at a specified interest rate for a specified period of time and then repaid at the end of that period, the amount repaid is

$\textit{Initial amount borrowed} + \textit{Amount of Interest for the time period}$, where

$\textit{Amount of Interest for the time period} = \textit{Amount borrowed} \times \textit{Interest rate for the time period}$.

Alternatively, the interest rate for the time period is equal to

$\dfrac{\textit{Amount of Interest for the time period}}{\textit{Amount borrowed at the start of the period}}$.

The "interest rate for the time period" mentioned above is generally found from the quoted annual interest rate and applied over a period of time that may not be one year. It is important to note that the phrase "quoted annual interest rate" must be more precisely defined in order to know how to apply it over a specified period of time.

**Simple Interest**

At **annual simple interest rate $\boldsymbol{i}$**, an initial investment of amount 1 accumulates to $\boldsymbol{1 + i \times t}$ at time $t$. In this expression, $t$ represents time and is usually measured in years. When simple interest at annual rate $i$ is specified, the interest rate for a period of time of length $t$ years is $i \times t$. If the amount of an investment or loan is $C$ (at time 0), then at time $t$ years, the amount of interest is $C \times i \times t$, and the total value of the investment or amount owing on the loan will be $C \times [1 + i \times t]$.

When simple interest is specified, there are variations in the way in which $t$ can be measured. The two most common forms of simple interest are the following.
(i) **Ordinary simple interest**, in which $t = \frac{m}{12}$, where $m$ is the number of months in the time period of the loan or investment. If a time period specifies a number of months in the context of simple interest, then this would be the way that $t$ is measured.
(ii) **Exact simple interest**, in which $t = \frac{d}{365}$, where $d$ is the exact number of days of the loan.
A variation of exact simple interest is the **Bankers Rule**, in which $t = \frac{d}{360}$.
Simple interest accumulation is usually restricted to periods of less than one year.

**Compound Interest**
At **annual effective compound interest rate** $\boldsymbol{i}$, an initial investment of 1 accumulates to $\boldsymbol{(1+i)^t}$ at time $t$ (years). Often, $t$ is a positive integer, but $t$ can also involve a fractional value with accumulation sometimes referred to as **true or exact compound interest**. Although the phrase "effective interest rate" usually refers to a rate which is compounded annually, an effective interest rate can also be specified for any period of time (such as monthly effective rate, quarterly effective rate, etc.) and compounding can be done accordingly, in which case, $t$ will be measured in appropriate units of time (months, quarters, etc.).

Note the following relationship between compound and simple interest for $i > 0$:
$(1+i)^t < 1 + it$ for $0 < t < 1$ and $(1+i)^t > 1 + it$ for $t > 1$.
For instance, with $i = .10$ (10%) and $t = 0.4$, we have
$(1.1)^{.4} = 1.03886 < 1.04 = 1 + (.4)(.1)$ .

Another point to note is that the phrase "annual effective interest rate" may also be seen in some references as "effective annual interest rate".

**Accumulation Function**
For an investment of 1 made at time 0, the value of the investment at time $t$ can be described in terms of an **accumulation function** $\boldsymbol{A(t)}$. For the case of simple interest at annual rate $i$ the accumulation function is $A(t) = 1 + it$, and for compound interest it is $A(t) = (1+i)^t$. The accumulated value (AV) is also called the **future value**, and can be applied to the combined value of several amounts at a particular time point in the future.

**Example 1 (SOA):** Money accumulates in a fund at an annual effective interest rate of $i$ during the first 5 years, and at an annual effective interest rate of $2i$ thereafter. A deposit of 1 is made into the fund at time 0. It accumulates to 3.09 at the end of 10 years and to 13.62 at the end of 20 years. What is the value of the deposit at the end of 7 years?

**Solution:** The accumulated value at time 10 is $(1+i)^5 \times (1+2i)^5 = 3.09$,
accumulated value at time 20 is $(1+i)^5 \times (1+2i)^{15} = 13.62$.
Then $\frac{(1+i)^5 \times (1+2i)^{15}}{(1+i)^5 \times (1+2i)^5} = (1+2i)^{10} = \frac{13.62}{3.09} = 4.4078 \rightarrow 2i = .16$, $i = .08$.
AV at the end of 7 years is $(1+i)^5 \times (1+2i)^2 = (1.08)^5 \times (1.16)^2 = 1.98$. □

**Example 2 (SOA):** Carl puts 10,000 into a bank account that pays an annual effective interest rate of 4% for ten years. If a withdrawal is made during the first five and one-half years, a penalty of 5% of the withdrawal is made. Carl withdraws $K$ at the end of each of years 4, 5, 6 and 7. The balance in the account at the end of year 10 is 10,000. Calculate $K$.

**Solution:** There are two (equivalent) ways to approach this problem. As a first approach, we can update the balance in the account at the time of each transaction until we reach the end of 10 years, and set the balance equal to 10,000 to solve for $K$ from the resulting algebraic expression:
balance at $t=4$ (after interest and withdrawal) is $B_4 = 10,000 \times (1.04)^4 - (1.05)K$ ;
balance at $t=5$ is $B_5 = B_4 \times 1.04 = [10,000(1.04)^4 - (1.05)K](1.04) - (1.05)K$ ;
at $t=6$ , $B_6 = [[10,000(1.04)^4 - (1.05)K](1.04) - (1.05)K](1.04) - K$ ;
at $t=7$ , $B_7 = [[[10,000(1.04)^4 - (1.05)K](1.04) - (1.05)K](1.04) - K](1.04) - K$ ;
at $t=10$, there is 3 years of compounding from time 7, so that $B_{10} = B_7 \times (1.04)^3$
$= [[[[10,000(1.04)^4 - (1.05)K](1.04) - (1.05)K](1.04) - K](1.04) - K](1.04)^3 = 10,000.$
Solving for $K$ from this equation results in $K = 979.93$.

Alternatively, we can accumulate to time 10 the initial deposit and the withdrawals separately, subtracting the accumulated withdrawals from the accumulated deposits. The balance at time 10 is
$10,000(1.04)^{10} - K(1.05)(1.04)^6 - K(1.05)(1.04)^5 - K(1.04)^4 - K(1.04)^3 = 10,000$ .
This is the same equation as in the first approach (and must result in the same value of $K$).

**In general, when using compound interest, for a series of deposits and withdrawals that occur at various points in time, the balance in an account at any given time point is the accumulated values of all deposits minus the accumulated values of all withdrawals to that time point.** This is also the idea behind the "dollar-weighted rate of return", which will be discussed in a later section of this study guide. □

**Present Value**

The **present value (PV) of 1 due in one year** is the amount required now to accumulate to amount 1 as of the end of one year from now. For instance, at an annual effective interest rate of 10%, $X$ invested now accumulates to $X \times 1.1$ one year from now. In order for this accumulated value to be 1, we must have $X \times 1.1 = 1$, or equivalently, $X = \frac{1}{1.1} = .9091$; this is the present value of 1 due in 1 year. In general, at annual effective rate of interest $i$, the present value of 1 due in one year is $\frac{1}{1+i}$, which is denoted $\boldsymbol{v_i}$ in actuarial notation. Often the subscript $i$ is dropped from the $v$-factor, if the interest rate being used is obvious from the context of the situation being considered. $v = \frac{1}{1+i}$ is sometimes referred to as the **present value factor** or **discount factor**. Note that if $i > 0$, then $v < 1$.

The **present value of 1 due in $t$ years** is the amount required now to accumulate to amount 1 as of the end of $t$ years from now. Present value is usually formulated in the context of compound interest, with the present value of 1 due in $t$ years equal to $\boldsymbol{\frac{1}{(1+i)^t} = (1+i)^{-t} = v_i^t}$ (or, more simply, $v^t$). If the compound period is a month or a quarter, we can still define a present value factor for the appropriate compounding period. For instance, with an monthly effective interest rate of 2% (compounding at 2% every month), the one-month present value factor (or discount factor) would be $\frac{1}{1.02} = v_{.02} = .9804$ (this is the amount needed now to accumulate to 1 in one month), and the present value of 1 due in $t$ months is $\frac{1}{(1.02)^t} = v_{.02}^t$ (this is the amount needed now to accumulate to 1 in $t$ months).

Present value can also be formulated on the basis of simple interest, where the present value of 1 due in $t$ years is equal to $\frac{1}{1+it}$. More generally, if an amount of 1 invested now accumulates to $A(t)$ in $t$ years, then the PV of 1 due in $t$ years based on that accumulation function is $\frac{1}{A(t)}$.

An **equation of value** for a financial transaction equates, at a particular point in time, the present and accumulated value of all amounts received with the present and accumulated values of all amounts to be disbursed (or paid out).

An investment will consist of cash outflows and inflows over time. There may a single large investment (outflow) at time 0 followed by income (inflows) in the future, or the investment may involve several outflows at different times. Suppose we denote by $C_k$ the net outflow/inflow at time point $k$, including time 0, where a negative value of $C_k$ indicates an outflow. If an investor has a target rate of return $i$ per time period for the investment, then the **net present value** of the investment based on that return is $\sum_{k=0}^{n} C_k \times v_i^k$. The net present value can be a basis for comparing investment options.

**Example 3 (SOA):** At an annual effective interest rate of $i$, $i > 0$, each of the following two sets of payments has present value $K$:

(i) A payment of 121 immediately and another payment of 121 at the end of one year.

(ii) A payment of 144 at the end of two years and another payment of 144 at the end of three years.

Calculate $K$.

**Solution:** At time 0 (now) the equation of value which equates the present values of the two sets of payments is $K = 121 + 121v = 144v^2 + 144v^3$. After factoring, this becomes $121(1+v) = 144v^2(1+v) \rightarrow v^2 = \frac{121}{144} \rightarrow v = \frac{11}{12} \rightarrow K = 121 + 121(\frac{11}{12}) = 231.92$.
Note that the value of $i$ implied is the solution of $\frac{11}{12} = v = \frac{1}{1+i}$, so that $i = \frac{1}{11} = .0909$. □

**Calculator Note 1, Accumulated and Present Value**

**Accumulated values and present values of single payments using annual effective interest rates can be made in the following way. Clear calculator registers before starting the keystroke sequence with [2nd] [CLR WORK].**

**Accumulated Value: A deposit of 100 made at time 0 grows at annual effective interest rate 5%. The accumulated value at the end of 10 years is $100(1.05)^{10} = 162.89$. This can be found using the calculator in two ways.**

**1. Calculator in standard-calculator mode.**
**Key in 1.05 [$y^x$], key in 10, [=] [×], key in 100, [=]**
**The screen should display 162.8894627. In this function, $y = 1.05$ and $x = 10$.**
**2. Calculator in prompted-worksheet mode.**
**Key in [2nd] [P/Y] [↓] key in 1 (this sets 1 compounding period per year).**
**Key in [2nd] [QUIT] (this returns calculator to standard-calculator mode)**
**Key in 100 [PV] [ENTER], key in 5 [I/Y] [ENTER],**
**key in 10 [N] [ENTER], key in [CPT] [FV]**
**The screen should display $-162.8894627$.**

**The calculator interprets the PV of 100 as an amount received (an in-cashflow) and the FV as the amount that must be paid back (an out-cashflow), so the FV is a "negative" cashflow. If the PV had been entered as $-100$, then FV would have been positive.**

**Present Value: The present value of 500 due in 8 years at annual effective rate of interest 8% is $500v_{.08}^{8} = 270.13$ . This can be found using the calculator in two ways.**

**1. Calculator in standard-calculator mode.**

**Key in 1.08 [ $y^x$ ], key in 8 , [ $+/-$ ] [ = ] [ × ], key in 500 [ = ]**

**The screen should display 270.1344423. This keystroke sequence can be replaced by**

**Key in 1.08 [ $1/x$ ] [ $y^x$ ], key in 8 , [ = ] [ × ], key in 500 , [ = ].**

**2. Calculator in prompted-worksheet mode.**

**Key in [ 2nd ] [ P/Y ] [ ↓ ] key in 1 (this sets 1 compounding period per year).**

**Key in [ 2nd ] [ QUIT ] (this returns calculator to standard-calculator mode)**

**Key in 500 [ FV ] [ ENTER ], key in 8 [ I/Y ] [ ENTER ],**

**key in 8 , [ N ] [ ENTER ], key in [ CPT ] [ PV ]**

**The screen should display $-270.1344423$. (same comment applies about the negative value).**

**As a more general procedure, in the equation $(PV)(1+i)^N = FV$ , if any 3 of the 4 variables PV , i , N , FV are entered, then the 4th can be found using the [ CPT ] function. As an example, suppose that an initial investment of 100 at annual effective rate of interest $i$ grows to 300 in 10 years. Then $100(1+i)^{10} = 300$ , from which we get $i = (3)^{1/10} - 1 = .1161$ (11.61%). The keystroke sequence solving for $i$ is**

**key in [ 2nd ] [ P/Y ] [ ↓ ] key in 1 ,**

**key in 100 [ PV ] [ ENTER ], key in 300 [ $+/-$ ] [ FV ] [ ENTER ],**

**key in 10 , [ N ] [ ENTER ] [ CPT ] [ I/Y ]**

**The screen should display 11.61 (this is the % measure).**

The function described in Calculator Note 1 is also valid for fractional exponents. For instance, $100 \times (1.05)^{10.5}$ can be found by keying in 10.5 instead of 10 in the accumulated value calculation above.

**Rate of Discount $d$**

Just as miles and kilometers are alternative measures for distance, there are alternative ways for measuring investment behavior. Actuarial terminology has a phrase to describe alternative ways of measuring investment growth. To say that two measures are **equivalent rates** of growth means that the measure describe the same pattern of investment growth.

The **annual effective rate of discount $d$** is a way of measuring investment growth but it is usually applied in the context of formulating present value: $1-d$ is the present value of 1 due in 1 year. We have already seen that given an annual effective interest rate of $i$. the present value of 1 due in one year is $v=\frac{1}{1+i}$. Now using an annual effective discount rate $d$, we can also formulate the present value of 1 due in one year as $1-d$. If we equate the two formulations we have for present value, we get $v=\frac{1}{1+i}=1-d$. If these relationships are satisfied, we say that $i$ and $d$ are **equivalent rates**, as they both describe the same pattern of investment behavior. For equivalent rates of interest $i$ and discount $d$, their numerical will generally not be the same. For instance, if $i=.10$ , then $v=\frac{1}{1.1}=.9091=1-.0909$, so in order for the discount rate $d$ to be equivalent to $i=.10$ the relationship $.9091=1-d$ must be satisfied, and therefore, $d=.0909$. Thus, the annual effective interest rate of 10% is equivalent to the annual effective discount rate of 9.09%. Just as a mile and a kilometer are not equal in length, neither are the annual effective interest rate and the annual effective discount rate equal numerically. Miles or kilometers can be used to describe equal lengths (one mile = 1.625 km) and interest and discount can be used to describe the same investment behavior: $i=.10$ and $d=.0909$ both refer to a PV factor of $.9091$, and they are referred to as equivalent rates.

**Although they are different numbers, $i$ and $d$ can refer to the same growth pattern for invested money. To say that .9091 grows to 1 in 1 year means that the money has grown by 10% (that is the interest rate $i$). It also means that the present value at the start of the year is 9.09% less than the accumulated value of 1 at the end of the year, which is described by saying that .9091 is the discounted value of 1 at discount rate 9.09%.**

Note the following relationships that come from equivalent $i$ and $d$:
$i$ and $d$ are equivalent rates (although different numerically) if $\boldsymbol{1-d=v=\frac{1}{1+i}}$ , or alternatively, if $\boldsymbol{d=\frac{i}{1+i}}$ or $\boldsymbol{i=\frac{d}{1-d}}$.
Thus, equivalent rates $i$ and $d$ satisfy the inequality $d<i$ (for positive rates).

Another way of looking at the discount rate and its connection the equivalent interest rate is as follows. It was mentioned earlier that the interest rate for the time period is equal to
$\frac{\textit{Amount of Interest for the time period}}{\textit{Amount borrowed at the start of the period}}$.
The discount rate for that same time period is $\frac{\textit{Amount of Interest for the time period}}{\textit{Amount owing at the end of the period}}$.
We can see this in very basic case in which the loan is of amount 1 with interest for the period at rate $i$. The interest rate is, or course $\frac{i}{1}$, or just $i$ for the period. The amount owing at the end of the period is $1+i$ and the discount rate is $\frac{i}{1+i}$, which we see is $d$ in the equivalence relationship mentioned in the previous paragraph. Again, $i$ and $d$ are different numerically. They are two different, but equivalent ways of describing the same transaction.

Note also that for equivalent $i$ and $d$, we have the relationship $id = i - d$, and $\frac{1}{d} = \frac{1}{i} + 1$. The algebraic link between interest and discount rates applies to any periodic rates. If $j$ is the compound rate of interest per month and $d_j$ is the equivalent compound rate of discount per month, then $\boldsymbol{v_j = \frac{1}{1+j} = 1 - d_j}$, $\boldsymbol{d_j = \frac{j}{1+j}}$ and $\boldsymbol{j = \frac{d_j}{1-d_j}}$.

Compound discount can also be formulated, with the present value of 1 due in $n$ years being $\boldsymbol{(1-d)^n = v^n}$. It is possible (although somewhat unnatural) to use compound discount to formulate accumulation. At annual effective discount rate $d$, the accumulated value of 1 in $n$ years is $(1-d)^{-n}$, since $\boldsymbol{(1-d)^{-1} = 1+i}$ for the equivalent annual effective rate of interest $i$. For instance, if the annual effective rate of discount is 6%, then the equivalent annual effective rate of interest is $\frac{.06}{1-.06} = .063830$ (6.3830%), and
$1 - .06 = .94 = \frac{1}{1.06383} = (1.06383)^{-1}$. The accumulation factor for 5 years would be $(1.063830)^5 = 1.3626 = (1-.06)^{-5} = (.94)^{-5}$.

An annual **simple discount rate** $\boldsymbol{d}$, refers to the situation in which the present value of 1 due in $t$ years is $1 - dt$, with similar considerations given to measuring $t$ as in the case of simple interest.

When an amount of 1 is invested at time 0 and the investment growth is represented by the accumulation function $A(t)$, the investment growth from time $t_1$ to time $t_2$ is $A(t_2) - A(t_1)$. This may also be described as the **amount of interest** earned from time $t_1$ to time $t_2$.

**Calculator Note 2 , Accumulated and Present Values Using a Discount Rate**

**Accumulated values and present values of single payments using an annual effective rate of discount can be made in the following way. Clear calculator registers before starting the keystroke sequence.**

**Accumulated Value: A deposit of 25 made at time 0 grows at annual effective discount rate 6%. The accumulated value at the end of 5 years is**
$25(1-.06)^{-5} = 25(.94)^{-5} = 34.06$.
**This can be found using the calculator in two ways.**
**Calculator is in standard calculator mode.**

**1. Key in .94 [ $y^x$ ], key in 5 [ +/− ] [ = ] [ × ], key in 25 , [ = ]**
**The screen should display 34.06 (rounded to nearest .01).**

**2. Key in 25 , [ PV ] [ ENTER ], key in 6 , [ +/− ] [ I/Y ] [ ENTER ],**
**key in 5 , [ +/− ] [ N ] [ ENTER ] [ CPT ] [ FV ]**
**The screen should display −34.06 (negative sign indicating outflow).**

**Present Value: The present value of 500 due in 8 years at annual effective rate of discount 8% is** $500(1-.08)^8 = 500(.92)^8 = 256.61$. **This can be found using the calculator in two ways.**

**Calculator is in standard calculator mode.**

**1. Key in .92 [ $y^x$ ], key in 8 , [ = ] [ × ], key in 500 , [ = ]**
**The screen should display 256.61.**

**2. Key in 500 , [ FV ] [ ENTER ], key in 8 , [ +/− ] [ I/Y ] [ ENTER ],**
**key in 8 , [ +/− ] [ N ] [ ENTER ] [ CPT ] [ PV ]**
**The screen should display −256.61.**

**Example 4 (SOA):** A deposit of $X$ is made into a fund which pays an annual effective interest rate of 6% for 10 years. At the same time, $X/2$ is deposited into another fund which pays an annual effective rate of discount of $d$ for 10 years. The amounts of interest earned over the 10 years are equal for both funds. Calculate $d$.

**Solution:** Amount of interest earned during ten years is equal to
accumulated value at time 10 $-$ initial amount invested :
the amount of interest earned by first fund is $X(1.06)^{10} - X = .790848X$
the amount of interest earned by the second fund is $\frac{X}{2}(1-d)^{-10} - \frac{X}{2} = \frac{X}{2}[(1-d)^{-10} - 1]$ .
Since the amount of interest earned over the 10 years is the same for both funds, it follows that
$.790848X = \frac{X}{2}[(1-d)^{-10} - 1] \rightarrow d = .0905.$ □

**Average Rate of Interest**

The conventional notion of the average of a set of numbers is the "simple average". This is found by adding the numbers and dividing the total by how many number are the set. When looking at compound interest over several compounding periods with differing effective rates for the periods, the notion used for average compound rate per period is not the simple average. For instance, suppose an investment of 1 grows to a value of $A(2)$ at time 2 years. If $i$ is the average annual effective rate of interest for each of the two years, then $A(2) = (1+i)^2$ would be used to find $i$. If the annual effective rates of interest in the first and second years were 5% and 15%, then $A(2) = 1.05 \times 1.15 = 1.2075 = (1+i)^2$, so that $i = .098863$. This is less than the simple average of 5% and 15%, which is 10%. The average rate $i$ is $\sqrt{1.05 \times 1.15}$ . This is called the "geometric average" of 1.05 and 1.15. In general, the average annual effective growth factor for a period of years is the geometric average of the growth factors for the various years:
$1 + i = [(1+i_1) \times (1+i_2) \times \cdots (1+i_n)]^{1/n}$ . In general, the geometric average of positive numbers is less than the simple average unless all the numbers are the same, in which case the geometric average is equal to the simple average.

# PROBLEM SET 1
## Effective Rates of Interest and Discount

1. (SOA) Gertrude deposits 10,000 in a bank. During the first year, the bank credits an annual effective interest rate of $i$. During the second year, the bank credits an annual effective rate of interest $(i-5\%)$. At the end of two years, she has 12,093.75 in the bank. What would Gertrude have in the bank at the end of three years, if the annual effective rate of interest were $(i+9\%)$ for each of the three years?

A) 16,851 B) 17,196 C) 17,499 D) 17,936 E) 18,113

2. (SOA) At an annual effective interest rate of $i$, $i>0$, the following are all equal:
(i) the present value of 10000 a the end of 6 years:
(ii) the sum of the present values of 6000 at the end of year $t$ and 56000 at the of year $2t$; and
(iii) 5000 immediately.
Calculate the present value of a payment of 8000 at the end of year $t+3$ using the same annual effective interest rate.

A) 1334 B) 1414 C) 1604 D) 1774 E) 2004

3. Smith invests \$10,000 in a 120-day short-term guaranteed investment certificate at a bank, based on exact simple interest at annual rate of 9.5%. After 60 days, the interest rate has risen to 12% and Smith would like to redeem the certificate early and reinvest in a 60-day certificate at the higher interest rate. In order for Smith to have no advantage in redeeming early and reinvesting at the higher rate, what early redemption penalty (deducted from the accumulated value of the investment certificate to that point) should the bank charge Smith at the time of early redemption (answer to nearest \$1)?

A) \$39 B) \$41 C) \$43 D) \$ 45 E) \$47

4. The managers of ABC Mutual Fund have reported an average annual effective return of 14% for the 10 years ending Dec. 31, 2004, and an average annual effective return of 16% for the 5 years ending Dec. 31, 2004. Find the average annual effective return for the 5 years ending Dec. 31, 1999.

A) 11% B) 11.5% C) 12% D) 12.5% E) 13%

**PROBLEM SET 1**

5. Which of the following statements are correct for equivalent rates $i$ and $d$?

I. $\lim_{i \to 0} d = 0$

II. $\lim_{i \to \infty} d = \infty$

III. $\frac{1}{d} - \frac{1}{i} = i$

IV. $(1-v)i = i - d$

V. $1 + it < (1+i)^t$ for any $i > 0$

VI. $1 - dt < (1-d)^t$ for any $d > 0$

6. Smith will invest \$200 today and \$100 one year from now. Suppose that this year's interest rate is $i_1$ and next year's rate will be $i_2$. Suppose that we define Smith's two-year average return will be $i$, where Smith's accumulated investment amount at the end of two years is $100[2(1+i)^2 + (1+i)]$.
Find $i$ under each of the following interest rate environments (assume $i > 0$):
(a) $i_1 = i_2 = .10$ (b) $i_1 = .08\,,\ i_2 = .10$ (c) $i_1 = .10\,,\ i_2 = .08$

7. You are given the following information.
Initial deposit to a fund: \$35,000
Withdrawal from the fund at the end of the fourth year: \$70,000
Value of the fund at the end of the eighth year: \$14,000
No other deposits or withdrawals were made during the eight-year period.
In what range is the annual rate of return for the fund during the eight-year period?
A) Less than 14% B) 14% but less than 19% C) 19% but less than 24%
D) 24% but less than 29% E) 29% or more

8. (SOA May 05) Mike receives cash flows of 100 today, 200 in one year, and 100 in two years. The present value of these cash flows is 364.46 at an annual effective rate of interest $i$.
Calculate $i$.
A) 10% B) 11% C) 12% D) 13% E) 14%

9. (SOA May 05) A store is running a promotion during which customers have two options for payment. Option one is to pay 90% of the purchase price two months after the date of sale. Option two is to deduct $X$% off the purchase price and pay cash on the date of sale.
A customer wishes to determine $X$ such that he is indifferent between the two options when valuing them using an annual effective interest rate of 8%. Which of the following equations of value would the customer need to solve?

A) $\left(\frac{X}{100}\right)\left(1+\frac{0.08}{6}\right)=0.90$ B) $\left(1-\frac{X}{100}\right)\left(1+\frac{0.08}{6}\right)=0.90$

C) $\left(\frac{X}{100}\right)(1.08)^{1/6}=0.90$ D) $\left(\frac{X}{100}\right)\left(\frac{1.08}{1.06}\right)=0.90$ E) $\left(1-\frac{X}{100}\right)(1.08)^{1/6}=0.90$

10. At an annual effective rate of interest of $i$, a payment of 100 right now combined with the present value of a payment of 100 to be made 3 years from now has same total present value right now as the present value now of a single payment of 190 to be made 1 year from now.
Find $i$.

# PROBLEM SET 1 SOLUTIONS

1. $10,000(1+i)(1+i-.05)=12,093.75 \rightarrow 10,000(1+i)^2-500(1+i)-12,093.75=0$ . Solving the quadratic equation results in $1+i=1.125$ , $-1.075$ . We ignore the negative root. With $i=.125$ we get $i+.09=.215$ . Then $10,000(1.215)^3=17,936$ .Answer: D

2. $5,000=10,000v^6=6,000v^t+56,000v^{2t} \rightarrow v^6=.5$ , and solving the quadratic in $v^t$ we get $v^t=.25$ . Then $8000v^{t+3}=8000(.25)(\sqrt{.5})=1414$. Answer: B

3. $10,000[1+\frac{120}{365}\cdot(.095)]=10,312.33$ is the maturity amount of the original GIC. At 60 days, the accumulated value (at 9.5%) is $10,000[1+\frac{60}{365}\cdot(.095)]=10,156.16$ . The amount required to invest for the next 60 days at 12% to reach the original maturity amount is $\frac{10,312.33}{1+\frac{60}{365}\cdot(.12)}=10,112.84$ . Penalty $=10,156.16-10,112.84=43.32$ . Answer: C

4. If average annual effective return (or rate of interest) is $i$ for the first 5 years, we have
$(1+i)^5(1.16)^5=(1.14)^{10} \rightarrow i=.1203$ .
Note that "average annual effective rate" for the period from time 0 to time $n$ years would result in a growth factor of $(1+i)^n$ for the $n$-year period. This average rate would usually be specified in the context of some other growth pattern over the $n$-year period to which the average rate is equivalent. Since the average rate in this case is $i$ for the first five years, the accumulation factor over the first five years would be $(1+i)^5$. This is then followed by 5 more years at an average annual rate of 16%, so the combined 10 year growth factor is $(1+i)^5(1.16)^5$. But since we are also told that the average rate for the 10 year period is 14%, we can formulate the 10 year growth factor as $(1.14)^{10}$. This gives us two equivalent ways of representing the 10 year growth factor. The equation of value is formulated by setting these two growth factors to be equal, after which we solve for $i$. Answer: C

5. Which of the following statements are correct for equivalent rates $i$ and $d$?
I. $d = \frac{i}{1+i} \to 0$ as $i \to 0$. True.
II. $d = \frac{i}{1+i} \to 1$ as $i \to \infty$. False.
III. $\frac{1}{d} - \frac{1}{i} = \frac{1+i}{i} - \frac{1}{i} = 1$. False
IV. $(1-v)i = i - iv = i - \frac{i}{1+i} = i - d$. True
V. $1 + it > (1+i)^t$ for any $i > 0$ and $0 < t < 1$. False.
VI. $1 - dt < (1-d)^t$ for any $d > 0$ and $t < 1$. False.

6. (a) $\text{AV} = 100[2(1.1)^2 + 1.1] = 100[2(1+i)^2 + (1+i)]$
$\to 2(1+i)^2 + (1+i) - 3.52 = 0 \to 1+i = 1.1, \ -1.6 \to i = .1$
(this could have been anticipated from the simple form of the equation).
(b) $\text{AV} = 100[2(1.08)(1.1) + 1.1] = 100[2(1+i)^2 + (1+i)]$
$\to 2(1+i)^2 + (1+i) - 3.476 = 0 \to 1+i = 1.0918$.
(c) $\text{AV} = 100[2(1.1)(1.08) + 1.08] = 100[2(1+i)^2 + (1+i)]$
$\to 2(1+i)^2 + (1+i) - 3.456 = 0 \to 1+i = 1.0881$.
Note that in both cases (b) and (c), $i$ is not .09, the "simple" average of .08 and .1. The rate $i_1$ affects growth over both a one-year and two-year period, but the rate $i_2$ affects growth only over the second year, so the average rate is more of "weighted average". $i_1$ has more "weight" than $i_2$ in the total accumulation over the two year period. From that point of view, it seems logical in case (b) that the average rate is greater than the simple average of .09, since more weight comes from $i_1$ which is larger than $i_2$. The same reasoning applies to case (c), since more weight comes from the $i_1$ which is now lower than $i_2$.

7. $35(1+i)^8 - 70(1+i)^4 = 14$. Let $x = (1+i)^4$, and the equation becomes
$5x^2 - 10x - 2 = 0$. Solving the quadratic equation results in
$x = \frac{10 \pm \sqrt{100+40}}{2(5)} = 2.183, \ -.183$. We discard the negative root, since $x = (1+i)^4 \geq 0$.
Then $(1+i)^4 = 2.183 \to i = .216$. Answer: C

8. $100 + 200v_i + 100v_i^2 = 364.46$ . We can solve for $i$ four ways.
(i) Trial and error. Try each interest rate until the PV is correct. We see that $i = .10$.
Note that this trial and error approach may be an efficient problem solving technique for certain problem types. In particular, if it is possible to reduce the solution of the problem to a single equation that involves the unknown variable, then it may be possible to substitute into that variable the various possible answers provided to see which one makes the equation correct.

(ii) Solve the equation $264.46 = 200v + 100v^2$ using the calculator financial functions with $PV = 264.46$ , $PMT = -200$ and $FV = 100$ .

(iii) Solve the equation $264.46 = 200v + 100v^2$ using the BA-II PLUS calculator cashflow spreadsheet.

(iv) Solve the quadratic equation $100v^2 + 200v - 264.46 = 0$ .
$v = \frac{-200 \pm \sqrt{200^2 - 4(100)(-264.46)}}{200} = .9091$ or $-2.9091$ (ignore the negative root).
$v = \frac{1}{1+i} = .9091 \rightarrow i = .10$ . Answer: A

9. Under Option one, the customer can pay $1 - \frac{X}{100}$ now. Under Option two the customer can pay .9 two months from now. These two options are equivalent in the sense that the customer is indifferent to picking one or the other. Therefore, the PV of a payment of .9 that would be made in two months is equal to a payment of $1 - \frac{X}{100}$ now. This can be expressed as
$1 - \frac{X}{100} = (.9)v_{.08}^{1/6}$ , or equivalently, $(1 - \frac{X}{100})(1.08)^{1/6} = .9$ . Answer: E

10. The equation of value as of now is $100 + 100v^3 = 190v$ .
We use the cashflow worksheet of the BA II PLUS calculator to solve this by setting
$CFo = 100$ , $C01 = -190$ , $F01 = 1$ , $C02 = 0$ , $F02 = 1$ , $C03 = 100$ , $F03 = 1$ .
Then calculate $IRR$ . The resulting interest rate is 17.21% .
Since this is a cubic equation, there are three solutions for $i$, but the calculator returns the smallest positive solution. The other solutions are 35.67% and a negative solution for $v$.

# SECTION 2 - NOMINAL RATES OF INTEREST AND DISCOUNT

**Sections 1.4-1.5 of "Mathematics of Investment and Credit"**

### Nominal Rate Of Interest

A typical credit card will describe the way in which interest is charged in something like the following way: "annual interest rate of 24% payable monthly". The phrase "payable monthly" is interpreted as meaning that the quoted annual rate of 24% is to be divided by 12 (the number of months in the year), resulting in 2%, and that is the rate charged each month on outstanding balances on the credit card account. If the 2% per month were to compound over 12 months, the effective growth for the year would be $(1.02)^{12} = 1.268242$, corresponding to an annual effective interest rate of 26.82% (rounded). If someone had an amount owing on the credit card account and if no additional purchases amounts were added to the account, then one year later the amount owing on the card account would have grown by a factor or 1.2682 (if the credit card issuer allowed the amount owing to continue for that long). The quoted annual rate of 24% is not an annual effective rate. In actuarial terminology, the 24% rate is referred to as a **nominal annual rate of interest**.

An interest rate that is quoted on an annual basis but is compounded based on a period of other than 1 year is referred to in actuarial terminology as a **nominal annual rate of interest**. The notation $\boldsymbol{i^{(m)}}$ is used to denote a **nominal annual rate of interest compounded (or convertible or payable)** $\boldsymbol{m}$ **times per year**. The notation $i^{(m)}$ implicitly refers to a $\frac{1}{m}$-year compounding period, and a $\frac{1}{m}$-year effective interest rate of $\frac{i^{(m)}}{m}$ (per $\frac{1}{m}$ years). For instance, in the credit card example mentioned above the quoted interest rate of 24% actually refers to a 1-month compounding period, with a one-month effective compound interest rate of $\frac{.24}{12} = .02$, or 2% (per month). Using $i^{(m)}$ notation in this example we have $i^{(12)} = .24$.

With an initial investment of amount 1 and a quoted nominal annual interest rate $i^{(m)}$, the $\frac{1}{m}$-year effective growth factor is $1 + \frac{i^{(m)}}{m}$. For an initial investment of 1, the accumulated value after $k$ successive $\frac{1}{m}$-ths of a year is $\left[1 + \frac{i^{(m)}}{m}\right]^k$ (this is the growth factor for that length of time), and after 1 year ($m$ successive $\frac{1}{m}$-ths) the accumulated value is $\left[1 + \frac{i^{(m)}}{m}\right]^m$. This would be the (effective) growth over a one year period. Note that it is almost always the case that $m$ is an integer; $m = 12$ corresponds to monthly compounding, $m = 4$ corresponds to quarterly compounding (every 3 months), etc.

**SECTION 2 - NOMINAL RATES OF INTEREST AND DISCOUNT**

We have seen that we can also represent the growth over a one year period as $1+i$, where $i$ is the annual effective interest rate. We have two alternative ways of describing annual effective growth. The notion of equivalent rates arises again, now in the context of the relationship between a nominal annual interest rate $i^{(m)}$ and an annual effective interest rate $i$. In order for an annual effective interest rate $i$ to be equivalent to the nominal annual interest rate $i^{(m)}$, the algebraic representation of one-year growth must be the same under both reprehensions, so the following equation must be satisfied: $\boldsymbol{1+i=\left[1+\frac{i^{(m)}}{m}\right]^m}$. This equation can be reformulated in the following way: $\boldsymbol{i^{(m)}=m\times[(1+i)^{1/m}-1]}$ .

Using the credit card example again, the nominal rate of 24% above is compounded monthly, and can be described in actuarial notation as $i^{(12)}=.24$. The annual effective growth is $(1+\frac{.24}{12})^{12}=(1.02)^{12}=1.2682$. We can describe this by saying that **a nominal annual rate of interest of 24% compounded monthly is equivalent to an annual effective rate of interest of 26.82%**. In the same way the notion of "equivalent rates" was used in the context of interest and discount rates, equivalent rates again means that an investment grows in the same way whether it is described as a nominal annual interest rate of 24% compounded monthly, or an annual effective interest rate of 26.82%.

Suppose that a nominal rate of interest is specified with compounding more than once per year. Once we determine the compounding period and compound rate, say $j$, per compounding period, we can then formulate accumulated and present values using $(1+j)^n$ and $\frac{1}{(1+j)^n}=v_j^n$ for $n$ compounding periods, including fractions of a compounding period. So, for instance, the nominal rate $i^{(12)}=.24$ corresponds to a one-month effective interest rate of 2% (with compounding every month), and the one-month compound present value factor would be $\frac{1}{1.02}=v_{.02}$.
We can formulate accumulated and present values using $(1.02)^n$ and $\frac{1}{(1.02)^n}=v_{.02}^n$ for $n$ months. When a nominal annual interest rate is being used, the important factors related to it are the number of compounding periods per year (and the corresponding length of the related compounding period) and the effective rate of interest for the compounding period. The reason for this is that when algebraic relationships are formulated for accumulation and present value, it is generally easiest and most efficient to represent them in terms of the compounding period. For instance, if a nominal annual rate compounded monthly $i^{(12)}$ is specified, it is usually best to find the monthly effective rate, say $j=\frac{i^{(12)}}{12}$, and formulate an equation of value in which time periods are measured in months and which uses $j$.

**Example 5:** Find the nominal annual rate of interest compounded every 6 months that is equivalent to an annual effective rate of interest of .268242.

**Solution:** The standard actuarial notation for the nominal annual rate compounded every 6 months is $i^{(2)}$, since there are $m = 2$ compounding periods per year. This refers to a 6-month compound rate of $\frac{i^{(2)}}{2}$. Suppose we denote this 6-month effective compound rate by $k$ for the time being. The effective growth for a year is $(1+k)^2$. We are told that the annual effective rate of interest is .268242, so the effective growth for the year is 1.268242. In order to have equivalence between these rates, we solve the equation $(1+k)^2 = 1.268242$, which results in $k = .126163$. This is the 6-month effective compound rate also denoted $\frac{i^{(2)}}{2}$. Therefore we have $\frac{i^{(2)}}{2} = .126163$ (.1262 after rounding) , and then $i^{(2)} = .2524$. This is the nominal annual rate of interest compounded every 6 months that is equivalent to the annual effective rate of interest of .2682. In general it is advisable to use the calculator accuracy for calculations, storing values and using them in subsequent calculations and avoid any rounding until reaching the final calculated number. In this case, this would result in $i^{(2)} = .2523$ after rounding to 4 decimal places. □

We can see in Example 5 the application of the relationship $i^{(m)} = m \times [(1+i)^{1/m} - 1]$, where $i = .2682$ is the annual effective rate and $m = 2$ is the number of compounding periods. Using this relationship give us $i^{(2)} = 2 \times [(1.2682)^{1/2} - 1] = .2522$, which is the equivalent nominal annual rate of interest compounded two time per year. Notice also that $\frac{i^{(2)}}{2} = (1.2682)^{1/2} - 1 = .1261$ is the equivalent 6-month effective rate of interest, and the nominal annual interest rate compounded twice per year $= 2 \times$ 6-month effective interest rate. In general, given an annual effective rate of interest $i$, the equivalent $\frac{1}{m}$-year effective interest rate is $(1+i)^{1/m} - 1$, and the equivalent nominal annual rate of interest is $m \times [(1+i)^{1/m} - 1]$ as seen above. Note also that $1 + \frac{i^{(m)}}{m} = (1+i)^{1/m}$, and $v^{1/m} = \frac{1}{1+\frac{i^{(m)}}{m}}$ .

The following points are not so likely to come up on the exam, but are worth mentioning.

(i) For $n > m > 1$, equivalent positive rates satisfy the inequality $i^{(n)} < i^{(m)} < i$.
For instance, in the discussion prior to Example 5 we have seen that nominal annual interest rate $i^{(12)} = .24$ (compounded monthly) is equivalent to annual effective interest rate $i = .2682$. In Example 5 we see that annual effective interest rate .2682 is equivalent to $i^{(2)} = .2524$ (compound twice per year). The numerical values are ordered as $.24 < .2524 < .2682$ , so in this case, for the equivalent rates $i^{(12)}, i^{(2)}$ and $i$, we have $i^{(12)} < i^{(2)} < i$ ($n = 12 > \ = 2 > 1$).

(ii) Although usually formulated in the context of $m$ as a positive integer, $i^{(m)}$ can be formulated for any $m > 0$. For instance, if we are told that the nominal annual rate of interest is 5% compounded once every three years, this means that $m = \frac{1}{3}$ , and $\frac{i^{(m)}}{m} = \frac{.05}{1/3} = .15$. The compounding period is 3 years, and the 3-year effective rate of interest is 15%; an investment of 1 grows to 1.15 in three years. The equivalent annual effective rate of interest would be found using $m = \frac{1}{3}$ from $i = \left[1 + \frac{i^{(m)}}{m}\right]^m - 1 = (1.15)^{1/3} - 1 = .0477$.

(iii) The symbol $s^{(m)}_{\overline{1}|i}$ denotes the factor $\frac{i}{i^{(m)}}$. A little more context for the meaning of this notation will be provided a little later in this Study Guide.

**The released FM exams have not referred to the $i^{(m)}$ notation very much. Questions almost always use verbal descriptions for describing interest rates. It is important to correctly interpret the language used to described interest rates. The following phrases all mean the same thing:**
(i) nominal annual interest rate of 24% compounded monthly,
(ii) nominal annual interest rate of 24% convertible monthly,
(iii) nominal interest rate of 24% compounded monthly ("annual" is implied),
(iv) nominal interest rate of 24% convertible monthly ("annual" is implied),
(v) annual interest rate of 24% compounded monthly ("nominal is implied),
(vi) annual interest rate of 24% convertible monthly ("nominal is implied).
If a question uses the phrase compounded monthly or convertible monthly or payable monthly (or quarterly, or semiannually, etc), the quoted rate is assumed to be a nominal annual rate of interest unless there is a clear indication otherwise. There is a considerable collection of terminology and notation that has been developed to describe and denote quantities that arise in actuarial mathematics. Exam questions occasionally refer to this notation, so it is important to be familiar with it, but more important is to understand that concepts represented by the notation.

**Calculator Note 3 , Conversion Between Equivalent Nominal and Annual Effective Rates**

**The nominal annual interest rate compounded $m$ times per year can be found from the annual effective rate of interest and vice-versa in the following way.**

**A nominal annual interest rate of .24 (24%) compounded monthly is equivalent to an annual effective rate of interest of $i = .2682$ (26.82%). The relationship**

**$i = (1 + \frac{i^{(12)}}{12})^{12} - 1$ can be used, or the equivalent rates can be found in the following way using the [ICONV] function.**

**1. Conversion from nominal annual to** annual **effective interest rate:**
**Key in [2nd] [ICONV] (NOM= appears) ,**
**key in 24 [ENTER] (the nominal rate) [↓] [↓] (C/Y= appears)**
**key in 12 [ENTER] (number of compounding periods) ,**
**key in [↓] [↓] (EFF= appears), press [CPT] .**
**The screen should display 26.82 . We have converted the nominal annual interest rate of 24% (key in 24) compounded monthly (key in 12) to the equivalent annual effective interest rate of 26.82% .**

**2. Conversion from** annual **effective to nominal annual interest rate:**
**Key in [2nd] [ICONV] [↓] (EFF= appears) ,**
**key in 26.82 [ENTER] [↓] (C/Y= appears),**
**key in 12 [ENTER] [↓] (NOM= appears) ,**
**key in [CPT] .**
**The screen should display 23.9966 (round to 24) . We have converted the annual effective interest rate of 26.82% (key in 26.82) to the equivalent nominal annual interest rate compounded monthly (key in 12) of 24% .**

The algebraic form of the relationship between the calculator entries "EFF" and "NOM" described in Calculator Note 3 is $EFF = (1 + \frac{NOM}{C/Y})^{C/Y} - 1 = (1 + \frac{.24}{12})^{12} - 1 = .2682$ , and $NOM = C/Y \times [(1 + EFF)^{1/(C/Y)} - 1] = 12 \times [(1.2682)^{1/12} - 1] = .24$ .
Therefore, if $NOM = i^{(m)}$ (the nominal annual rate) and $C/Y = m$ (the number of compounding periods per year) , then $EFF = (1 + \frac{i^{(m)}}{m})^m - 1 = i$ .
Note that "$C/Y$" is not a fraction, it is the number of compounding periods per year.
Although it may be awkward to do so, nominal interest rates can be defined in the case where $m$, the number of compounding periods per year is not an integer. $m$ can be less than 1 or a greater than 1. In the comments above, an example was given in which $m = \frac{1}{3}$. Example 8 below provides an example in which $m > 1$ but $m$ is not an integer. The calculator functions for the relationship between nominal and annual effective interest rates outlined in Calculator Note 3 above apply when $m$ is not an integer.

## SECTION 2 - NOMINAL RATES OF INTEREST AND DISCOUNT

A situation that arises regularly in exam questions is one in which an interest rate is given that has a particular compounding period, but the question requires an interest rate based on a different compounding period. This most often arises in the context of annuity valuation, and Section 6 of the study guide looks at this in detail. In general, given an interest rate for a particular compounding period, we can find the equivalent rate of interest for any other compounding period using the principle of "compound equivalence". For instance, if we are given an interest rate of .4% per month and we wish to have a rate which is compounded every 6 months, we just find $(1.004)^6 = 1.02424$, and we see that the equivalent rate of interest compounded every 6 months is 2.424%. We could have described the .4% per month rate of interest in an equivalent way as a nominal rate of interest of 4.8% compounded monthly, and we could have described the 2.424% rate of interest per 6-month period as a nominal rate of interest of 4.848% compounded semiannually. These are alternative ways of describing the same growth rate. It was not necessary to refer to the $i^{(m)}$ notation to describe these rates. The next example is similar to the one just considered and refers to the actuarial notation.

**Example 6:** Suppose that the nominal annual interest rate compounded 12 times per year is 18%. Find the equivalent annual effective rate of interest and the equivalent nominal annual interest rate compounded twice per year.

**Solution:** We are given $i^{(12)} = .18$, which is a nominal annual rate of interest of 18% compounded $m = 12$ times per year (also referred to as "compounded monthly"). This nominal rate refers to a one month compound rate of interest of $\frac{i^{(12)}}{12} = \frac{.18}{12} = .015$ (1.5% per month, compounded monthly). The accumulation factor for a full year would be $(1.015)^{12} = 1.1956$, and this corresponds to an equivalent annual effective interest rate of $i = .1956$.

Alternatively, we can use the relationship which links equivalent nominal and annual effective rates of interest : $i = \left[1 + \frac{i^{(12)}}{12}\right]^{12} - 1 = \left[1 + \frac{.18}{12}\right]^{12} - 1 = .1956$.

Once the annual effective rate of interest is known, the equivalent $i^{(2)}$ can be found from the relationship $i^{(2)} = 2 \times [(1+i)^{1/2} - 1] = 2 \times [(1.1956)^{1/2} - 1] = .1869$. This nominal annual rate compounded semi-annually refers to a 6-month compound rate of $\frac{i^{(2)}}{2} = \frac{.1869}{2} = .0935$.

Note that the numerical values of the equivalent $i$, $i^{(2)}$ and $i^{(12)}$ are different ($i > i^{(2)} > i^{(12)}$), but they are equivalent in the sense that they all describe the same accumulation pattern over a one year (or any) period. Note also that these calculations can be done quite efficiently in the BA II PLUS "interest conversion worksheet". This worksheet is opened with the keystrokes

[2nd] [ICONV]. The display should open with NOM= , and enter 18 (this is %) using keystrokes 18 [ENTER] ; then the keystrokes [↓] [↓] give you C/Y=, and you enter 12 (compounding periods); finally, the keystrokes [↓] [↓] give you EFF=, and you use the [CPT] key to get the annual effective rate of interest (it should display as 19.5618). To get the equivalent nominal rate convertible semi-annually, leave the EFF rate as it now is and use the [↓] key once to get C/Y= (it should display 12). Key in 2 [ENTER], then [↓] to get to NOM= (it should still display 18). Use the [CPT] key, and the display should change to 18.6886 , which is the equivalent nominal annual rate of interest convertible semiannually.
When working within a BA II PLUS worksheet, values change either when they are entered, or when they are computed based on the other entered values. □

**Example 7 (SOA):** Brian and Jennifer each take out a loan of $X$. Jennifer will repay her loan by making one payment of 800 at the end of year 10. Brian will repay his loan by making one payment of 1120 at the end of year 10. The nominal rate compounded semi-annually being charged to Jennifer is exactly one-half the nominal semi-annual rate being charged to Brian. Calculate $X$.
**Solution:** Let $j$ denote the six-month rate on Jennifer's loan (this would be one-half of the nominal annual rate on Jennifer's loan), so that the six-month rate on Brian's loan is $2j$. The equations of value of the two loans are
Jennifer : $X(1+j)^{20} = 800$ , Brian : $X(1+2j)^{20} = 1120$ .
Then, $\frac{X(1+2j)^{20}}{X(1+j)^{20}} = \frac{1120}{800} \rightarrow \frac{1+2j}{1+j} = (1.4)^{.05} = 1.01697 \rightarrow j = .01726$ .
$X = \frac{800}{(1+j)^{20}} = 568.13$ . Note that it was not necessary to use the $i^{(m)}$ notation to describe rates. □

**Example 8:** Smith opens up a short-term deposit contract at a bank. Smith deposits \$1000, and will receive \$1060 in 8 months. Express the interest rate paid on the deposit as a nominal annual rate $i^{(m)}$, and find the appropriate value of $m$.
**Solution:** The effective interest period is 8 months ($\frac{2}{3}$ of a year). In the context of a nominal annual rate of interest $i^{(m)}$, the effective period is $\frac{1}{m}$ years. Therefore, $\frac{1}{m} = \frac{2}{3}$, or equivalently, $m = \frac{3}{2}$. Also, the rate for the $\frac{1}{m}$ year period is $\frac{i^{(m)}}{m}$, so that $\frac{i^{(3/2)}}{3/2} = .06$ is the rate for $\frac{2}{3}$ of a year period. It then follows that $i^{(3/2)} = .09$. The effective period is $\frac{2}{3}$ of a year, so there are

$\frac{3}{2} = 1.5$ compounding periods per year. In previously released SOA exams on this topic, there has been infrequent reference to fractional $m$ in the context of nominal rates of interest. □

**Nominal Rate Of Discount**

Suppose that the discount rate for a 3-month period is .0225 . The present value factor for the 3-month period is $1 - .0225 = .9775$. The present value for a full year would be found by compounding this 3-month present value factor four times, so $v = (.9775)^4 = .9130$. This is the one-year effective present value factor, so we can find the equivalent annual effective discount rate $d$ from $.9130 = 1 - d$, so that $d = .0870$. We can also find the equivalent annual effective rate of interest, from $v = .9130 = \frac{1}{1+i}$, so that $i = .0953$; alternatively, from $d = .0870$ we get the equivalent annual effective interest rate from $i = \frac{d}{1-d} = \frac{.0870}{1-.0870} = .0953$.

The symbol $\boldsymbol{d^{(m)}}$ denotes a **nominal annual rate of discount compounded (or convertible) $\boldsymbol{m}$ times per year**, and it refers to a $\frac{1}{m}$-year compounding period, and a $\frac{1}{m}$-year effective discount rate of $\frac{d^{(m)}}{m}$ (per $\frac{1}{m}$ years). The present value of 1 due in $k$ successive $\frac{1}{m}$-ths of a year is $\left[1 - \frac{d^{(m)}}{m}\right]^k$, and the present value of 1 due in one year ($m$ successive $\frac{1}{m}$-ths) is $\left[1 - \frac{d^{(m)}}{m}\right]^m$.

In order for annual effective discount rate $d$ to be equivalent to nominal annual discount rate $d^{(m)}$, we must have

$$\boldsymbol{1 - d = \left[1 - \frac{d^{(m)}}{m}\right]^m}, \qquad \text{or equivalently,} \qquad \boldsymbol{d^{(m)} = m \times \left[1 - (1-d)^{1/m}\right]} \ .$$

In the numerical example described about, we have $m = 4$, and the 3-month discount rate of .0225 can be described as $\frac{d^{(4)}}{4} = .0225$, and the nominal annual discount rate compounded 4 times per year is $d^{(4)} = .09$.

Some of the comments made above about relationships involving equivalent nominal interest rates have counterparts for equivalent nominal discount rates.

(i) For $n > m > 1$ , equivalent (positive) rates satisfy the inequality $d < d^{(m)} < d^{(n)}$ ;

(ii) Although usually formulated in the context of $m$ being a positive integer, $i^{(m)}$ and $d^{(m)}$ can be formulated for any $m > 0$ .

 

(iii) For $n > m > 1$, we have the inequalities $d < d^{(m)} < d^{(n)} < i^{(n)} < i^{(m)} < i$ .

(iv) Some additional relationships are $\frac{1}{d^{(m)}} - \frac{1}{i^{(m)}} = \frac{1}{m}$ ; $i^{(m)} - d^{(m)} = \frac{i^{(m)} \cdot d^{(m)}}{m}$

It has often been the case in the SOA/CAS exam questions that the $i^{(m)}$ and $d^{(m)}$ notation is not used, but instead a nominal interest or discount is described verbally.

**Calculator Note 4 , Conversion Between Nominal Discount and Annual Effective Rates**

**The nominal annual discount rate compounded $m$ times per year can be found from the annual effective rate of interest and vice-versa in the following way.**

**A nominal annual discount rate of .09 (9%) compounded quarterly is equivalent to an annual effective rate of interest of $i = .0953$ (9.53%). The relationship $i = (1 - \frac{d^{(4)}}{4})^{-4} - 1$ can be used, or the equivalent rates can be found in the following ways.**

**Use the keystroke sequence [2nd] [P/Y] , 4 [ENTER] [2nd] [QUIT] (this sets the number of compounding periods per year to 4 and returns to standard calculation functions). It is important to do this step first, entering the number of compounding periods in the year.**

**1. We first find the equivalent annual effective rate of interest from the given nominal annual rate of discount.**
**Key in 4 [+/−] [N] [ENTER] , key in 9 [+/−] [I/Y] [ENTER] ,**
**key in 1 [PV] [ENTER] , key in [CPT] [FV] .**
**Calculator Note 4, continued**
**The display should read $-1.0953$ ; we interpret this is indicating that the annual effective rate of interest is 9.53% .**

**2. We now find the equivalent nominal annual rate of discount from the given annual effective rate of interest.**
**Key in 4 [+/−] [N] [ENTER] , key in 1.0953 [+/−] [FV] [ENTER] ,**

**key in 1 [PV] [ENTER] , key in [CPT] [I/Y] .**
**The display should read $-9.00$ ; this is the negative of the equivalent nominal annual rate of discount compounded 4 times per year.**
**Note that when we enter FV, we enter $-$ (1+EFF) .**

When a nominal rate of interest or discount is quoted and the objective is to calculate accumulated or present values, it is usually most efficient to find the interest rate or discount rate per compounding period, and then apply compounding based on the number of compounding periods.

**Example 9 (SOA):** At time $t = 0$, John deposits 1000 into a fund which credits interest at a nominal interest rate of 10% compounded semiannually. At the same time, he deposits $P$ into a different fund which credits interest at a nominal discount rate of 6% compounded monthly. At time $t = 20$, the amounts in the funds are equal. What is the annual effective interest rate earned on the total deposits, $1000 + P$, over the 20-year period?
**Solution:** At the end of 20 years, the amount in the first fund is $1000(1.05)^{40} = 7,039.99$ (the nominal interest rate of 10% compounded semiannually refers to a six-month compound rate of .05, and there are 40 six-month periods in 20 years). At $t = 20$, the amount in the second fund is $P(1 - .005)^{-240} = 3.3301P$ (the nominal discount rate of 6% compounded monthly refers to a one-month discount rate of .005, and there are 240 monthly periods in 20 years). We are given that the amounts in the two funds are equal at the end of 20 years, so that $3.3301P = 7,039.99$, from which we can solve for $P$; $P = 2114.05$ . The amount in each fund at the end of 20 years is $7,039.99$, so that the total in both funds is $14,079.98$. The total amount initially invested is $1000 + P = 3114.05$. The annual effective interest rate earned on the total deposits over the 20 year period is $i$, where $(3114.05)(1 + i)^{20} = 14,079.98$. Solving for $i$ results in $i = .0784$. Note that $i$ could also be viewed as the *average* annual effective interest rate on the combined deposits for the 20 year period. □

**U.S. And Canadian Treasury Bills**

One of the ways in which governments raise money for ongoing expenses is by borrowing from the public. A Treasury Bill (T-Bill) is a debt instrument used by governments for short-term (up to one year) borrowing. U.S. T-Bills are issued regularly with 13-week to 52-week maturity times. Canadian T-Bills are usually issued with 14-week, 24-week and 50-week maturities. T-Bills are sold to financial institutions as well as individuals, and there is a large secondary market for T-Bills. T-Bills are usually issued in billions of dollars of face amount.

U.S. T-Bills: In the U.S., T-Bill price and rate quotations follow certain conventions. The following example will illustrate this. The U.S. Treasury Department website for institutional investors listed the following T-Bill quotation:

| Term | Issue Date | Maturity Date | High Rate | Investment Rate | Price per $100 |
|---|---|---|---|---|---|
| 26-Week | 10/13/2016 | 04/13/2017 | 0.495% | 0.503% | 99.749750 |

The maturity date is 182 days, or 26 weeks after the issue date. This quotation format always assumes a maturity value of $100 on the maturity date, meaning that the government will pay $100 to the owner of the T-Bill on April 13, 2017. This quotation also tells us that for a T-Bill with face amount $100 maturing on April 13, 2017, the price is $99.749750 on October 13, 2016 when the T-Bill was issued. We can think of the price on the issue date as an amount discounted from the maturity value of $100. The discounted (or present) value represented with simple discount is $99.7489750 = 100 \times (1 - d \times t)$. In this equation, the discount rate $d$ is the "High Rate" in the listing above. $t$ represents the time, in years, until maturity, and the convention used for $t$ for U.S. T-Bills is $t = \frac{d}{360}$, where $d$ is the number of days to maturity. The measure of $t$ is based on 360 days in a year. In this case, $d = 182$ days, so that $t = .505556$, and the issue date price of the T-Bill is $100 \times (1 - .00495 \times \frac{182}{360}) = 99.749750$. The "Investment Rate" is the annual interest rate measure that is equivalent to this valuation with simple interest applied for the 182-day period in a 365-day year. Simple interest at rate $i$ for 182 days would have a present value factor of $\frac{1}{1+i\times\frac{182}{365}}$. In order to represent the T-Bill price using this simple interest form, we have $99.749750 = 100 \times \frac{1}{1+i\times\frac{182}{365}}$ .

Solving for $i$ from this relationship results in $i = .00503$. The Investment Rate may also be referred to as the yield rate or the "coupon equivalent", depending upon where the quotation is found. The Wall Street Journal will give quotations online for the secondary market in T-Bills. In that quote the price is not mentioned at all. What is provided is the maturity date, the (simple) discount rate from quotation date to maturity and the equivalent yield rate.

Canadian T-Bills: For quotation of interest rates Canadian T-Bills used simple interest only and a 365-day year to formulate the yield rate. As an example, an October 6, 2016 issue of a T-Bill maturing on March 23, 2017 (168 days, or 14 weeks) had a quoted yield of .530% and price of 99.75665 (per 100 of maturity amount). We can check that $99.75665 = 100 \times \frac{1}{1+.00530\times\frac{168}{365}}$.

Note that the formulation for yield rate is the same for both U.S. and Canadian T-Bills. The US uses the discount rate convention in addition to the yield rate.

# PROBLEM SET 2
## Nominal Rates of Interest and Discount

1. (SOA) Jennifer deposits 1000 into a bank account. The bank credits interest at a nominal annual rate of $i$ convertible semi-annually for the first 7 years and a nominal annual rate of $2i$ convertible quarterly for all years thereafter. The accumulated amount in the account at the end of 5 years is $X$. The accumulated amount in the account at the end of 10.5 years is 1980. Calculate $X$.
A) 1201  B) 1226  C) 1251  D) 1276  E) 1301

2. (SOA) You are given:
(i) Fund X accumulates at an interest rate of 8% compounded quarterly;
(ii) Fund Y accumulates at an interest rate of 6% compounded semiannually;
(iii) at the end of 10 years, the total amount in the two funds combined is 1000; and
(iv) at the end of 5 years the amount in Fund X is twice that in Fund Y.
Calculate the total amount in the two funds at the end of 2 years (nearest 10).
A) 560  B) 570  C) 580  D) 590  E) 600

3. (SOA) Calculate the nominal rate of interest convertible once every four years that is equivalent to a nominal rate of discount convertible quarterly.
A) $0.25[1-(1-0.25d^{(4)})^{-16}]$  B) $(1-0.25d^{(4)})^{-16}-1$  C) $(1-0.25d^{(4)})^{-8}-1$
D) $0.5[(1-0.25d^{(4)})^{-8}-1]$  E) $0.25[(1-0.25d^{(4)})^{-16}-1]$

4. Smith invests \$10,000 into a fund earning a nominal annual rate of interest of 8% compounded every 3 months. Jones invests \$10,000 into a fund earning a nominal annual rate of discount of 8% compounded every month. $I_S$ and $I_J$ denote the amounts of interest earned on Smith's and Jones' funds, respectively, in the last 6 months of the second year of their investments. Find $I_S/I_J$.
A) .985  B) .990  C) .995  D) 1.000  E) 1.005

5. (EA1) You are given that $1000d^{(m)}=85.256$ and $1000d^{(2m)}=85.715$, and they are equivalent rates. Find $i^{(3m)}$.
A) .0825  B) .0835  C) .0845  D) .0855  E) .0865

**PROBLEM SET 2**

6. (SOA) Calculate the nominal rate of discount convertible monthly that is equivalent to a nominal rate of interest of 18.9% per year convertible monthly.
A) 18.0% B) 18.3% C) 18.6% D) 18.9% E) 19.2%

7. (SOA) At a nominal interest rate of $i$ convertible semi-annually, an investment of 1000 immediately and 1500 at the end of the first year will accumulate to 2600 at the end of the second year. Calculate $i$.
A) 2.75% B) 2.77% C) 2.79% D) 2.81% E) 2.83%

8. Smith invests $1,000 in a 5-year "step-up" Guaranteed Investment Certificate (GIC). The GIC will pay interest, compounded monthly, at the following nominal annual rates:
Year 1: 3.0% , Year 2: 4.0% , Year 3: 6.0% , Year 4: 9.0% , Year 5: 12.0%
(a) Find Smith's average nominal annual rate of return compounded monthly over the 5-year period.
(b) One of the details of the GIC arrangement is that Smith can end the GIC early with a penalty. Suppose that Smith ends the GIC right at the beginning of the 5th year.
Find Smith's average annual effective return in each of the following cases:
(i) the penalty is 5% of the total interest earned on the investment up to the time it ended, and
(ii) the penalty is 5% of the value of the investment at the time it is ended.

9. (SOA) Let $S$ be the accumulated value of 1000 invested for two years at a nominal annual rate of discount $d$ convertible semiannually, which is equivalent to an annual effective interest rate of $i$. Let $T$ be the accumulated value of 1000 invested for one year at a nominal annual rate of discount $d$ convertible quarterly,. You are given that $S/T = (39/38)^4$. Calculate $i$.
A) 10.0% B) 10.3% C) 10.8% D) 10.9% E) 11.1%

10. Verify that for equivalent $i^{(m)}$ and $d^{(m)}$, the ratio $i^{(m)}/d^{(m)}$ is equal to $1 + \frac{i^{(m)}}{m}$.
Provide a brief verbal interpretation of this relationship.

11. A T-Bill is quoted at a price of $99.1215 per $100 of maturity amount. The T-Bill will mature in 13 weeks (91 days).
(a) Formulate the quoted annual discount rate for this T-Bill using the US convention.
(b) Formulate the quoted annual yield or investment rate for this T-Bill.
(c) If the interest found in part (b) is expressed as a nominal annual interest rate $i^{(m)}$, what would $m$ be?

12. In recent years "Payday Loan" has been coined to refer to a short term loan for an amount that is usually no more than $1500. The interest rates charged on these loans are generally high. An October, 2016 internet search turned up the following loan details:
- $100 loan to be repaid in 14 days
- Loan cost of $15
- APR (annual percentage rate) of 391.07%

Consumer loans in the U.S. must conform to the 1968 Truth In Lending legislation that requires that the APR for the transaction is provided. The APR is the nominal annual rate of interest equivalent to the rate being charged for the transaction.
For the loan described above, determine the APR and the equivalent annual effective rate of interest.

# PROBLEM SET 2 SOLUTIONS

1. $1000(1+\frac{i}{2})^{14}(1+\frac{2i}{4})^{14}=1980 \rightarrow (1+\frac{i}{2})^{28}=1.980 \rightarrow i=.0494$ .
$X=1000(1+\frac{i}{2})^{10}=1276$ . Answer: D

2. Suppose that Fund X starts with amount $A$ and Fund Y starts with amount $B$.
At end of 10 years: $A(1.02)^{40}+B(1.03)^{20}=1000$
At the end of 5 years: $A(1.02)^{20}=2B(1.03)^{10}$ .
Solving the two equations for $A$ and $B$ results in $A=311.86$ , $B=172.41$ .
At the end of 2 years: $A(1.02)^{8}+B(1.03)^{4}=559.45$ . Answer: A

3. The nominal rate of interest convertible once every four years is $i^{(1/4)}$.
The nominal rate of discount convertible quarterly is $d^{(4)}$. We use the relationships linking these rates to equivalent annual effective rates of interest:
$1+i=(1+\frac{i^{(m)}}{m})^{m}$ , $1+i=(1-\frac{d^{(n)}}{n})^{-n}$ .
Then $(1+\frac{i^{(1/4)}}{1/4})^{1/4}=(1-\frac{d^{(4)}}{4})^{-4}$ . Solving for $i^{(1/4)}$ we get
$i^{(1/4)}=\frac{1}{4}[(1-\frac{d^{(4)}}{4})^{-16}-1]=0.25[(1-0.25d^{(4)})^{-16}-1]$ . Answer: E

4. $I_S=10,000[(1.02)^{8}-(1.02)^{6}]=454.97$ .
$I_J=10,000[(1-.006667)^{-24}-(1-.006667)^{-18}]=461.92$ .
$I_S/I_J=.985$ Answer: A

5. Annual effective PV factor is $(1-\frac{d^{(2m)}}{2m})^{2m}=(1-\frac{d^{(m)}}{m})^{m} \rightarrow (1-\frac{.085715}{2m})^{2}=1-\frac{.085256}{m}$
$\rightarrow \frac{-.17143}{2m}+\frac{.085715^2}{4m^2}=\frac{-.085256}{m} \rightarrow m=4$ .
Annual effective growth is $(1+\frac{i^{(3m)}}{3m})^{3m}=(1-\frac{d^{(m)}}{m})^{-m} \rightarrow 1+\frac{i^{(3m)}}{3m}=(1-\frac{d^{(m)}}{m})^{-1/3}$
$\rightarrow i^{(3m)}=3m[(1-\frac{.085256}{4})^{-1/3}-1]=.086488$ . Answer: E

6. The monthly interest rate is $\frac{.189}{12}=.01575$ . The monthly discount rate $j$ is related to the monthly interest rate through the relationship $j=\frac{.01575}{1.01575}=.015506$.
The nominal annual discount rate convertible monthly is $12\times.015506=.1861$. Answer: C

7. The accumulate value at the end of the 2nd year of the two payments is
$1000(1+\frac{i}{2})^4 + 1500(1+\frac{i}{2})^2$ , which we are told is equal to 2600.
If we let $1+j = (1+\frac{i}{2})^2$ , then this equation becomes $1000(1+j)^2 + 1500(1+j) = 2600$ .
The cashflow worksheet on the BA-II PLUS can be used to solve for $j$, $j = .028342$ .
Alternatively, we can solve the quadratic equation and ignore the negative root.
Then $1.028342 = (1+\frac{i}{2})^2$ , so that $i = .0281$ . Answer: D

8. (a) At the end of 5 years, the value of the GIC is
$1000(1+\frac{.03}{12})^{12}(1+\frac{.04}{12})^{12}(1+\frac{.06}{12})^{12}(1+\frac{.09}{12})^{12}(1+\frac{.12}{12})^{12} = 1,403.28.$
This corresponds to an average monthly rate of return of $j$ over the 5-year period, where
$(1+j)^{60} = 1.40328$, so that $j = .005663$. This corresponds to a nominal annual rate of return compounded monthly of $12j = .0680$.
(b) The value of the investment before penalty at the beginning of the 5th year is
$1000(1+\frac{.03}{12})^{12}(1+\frac{.04}{12})^{12}(1+\frac{.06}{12})^{12}(1+\frac{.09}{12})^{12} = 1,245.34.$
(i) The interest earned over the 4 years is 245.34, and 5% of that is a penalty of 12.27.
The value of the GIC after penalty is $1,245.34 - 12.27 = 1,233.07.$
This corresponds to an average annual effective rate of return of $i$ over the 4-year period, where
$(1+i)^4 = 1.23307$, so that $i = .0538$.
(ii) The penalty is $1,245.34 \times .05 = 62.27$, and the value after penalty is
$1,245.34 - 62.27 = 1,183.07$.
This corresponds to an average annual effective rate of return of $i$ over the 4-year period, where
$(1+i)^4 = 1.18307$, so that $i = .0429$.

9. Notice that the $d^{(m)}$ notation is not used in this question although it involves nominal discount rates. $S = (1-\frac{d}{2})^{-4}$ and $T = (1-\frac{d}{4})^{-4}$ , so that $S/T = \left(\frac{1-\frac{d}{2}}{1-\frac{d}{4}}\right)^{-4} = \left(\frac{39}{38}\right)^4$ .
Therefore, $\frac{1-\frac{d}{4}}{1-\frac{d}{2}} = \frac{39}{38}$, from which we find $d = 0.1$ . The equivalent annual effective interest rate $i$ satisfies the relationship $1+i = (1-\frac{.1}{2})^{-2} = 1.10801$ . Answer: C

10. $1 - \frac{d^{(m)}}{m}$ is the PV of 1 due in $\frac{1}{m}$ years, which is also equal to $\frac{1}{1+\frac{i^{(m)}}{m}}$. Solving for $d^{(m)}$ results in $d^{(m)} = \frac{i^{(m)}}{1+\frac{i^{(m)}}{m}}$, or equivalently, $\frac{d^{(m)}}{m} = \frac{\frac{i^{(m)}}{m}}{1+\frac{i^{(m)}}{m}}$. This relationship represents the fact that for equivalent rates, the amount of discount at the beginning of a period is the PV of the amount of interest at the end of a period. In this case, the period is $\frac{1}{m}$ years. For the annual effective case we have $d = \frac{i}{1+i}$.

**PROBLEM SET 2**

11.(a) $99.1215 = 100 \times (1 - d \times \frac{91}{360})$. Solving for $d$ results in $d = .0348$ (rounded from .03475385).

(b) $99.1215 = \frac{100}{1+i\times\frac{91}{365}}$ → $i = .0355$ (rounded from .03554883).

(c) $m = \frac{91}{365}$ .

12. The interest rate on the 14-day loan is 15%. This is equivalent to an APR of $.15 \times \frac{365}{14} = 3.9107$, or 391.07%. The equivalent annual effective rate of interest is $(1.15)^{365/14} = 38.2366$, or 3,823.66%.

# SECTION 3 - FORCE OF INTEREST, INFLATION AND RISK OF DEFAULT

**Sections 1.6-1.7 of "Mathematics of Investment and Credit"**

**Force of interest**

We saw in the previous section that if the annual effective rate of interest is 26.8242%, then the equivalent nominal annual rate of interest compounded semiannually is 25.24%. This says that $i^{(2)} = .2524$ and $(1 + \frac{.2524}{2})^2 = (1.1262)^2 = 1.268242$. This is the same as saying that the equivalent rate of interest compounded every 6 months is 12.62%. We also saw that the equivalent nominal annual rate of interest compounded monthly was $i^{(12)} = .24$, meaning $(1 + \frac{.24}{12})^2 = (1.02)^{12} = 1.268242$, and the equivalent rate of interest compounded every month is 2%.

If we are given an annual effective rate of interest, then for any fraction of a year we can find an equivalent interest rate that is compounded based on that fraction of a year, and then we can describe it as an equivalent nominal annual rate. If the annual effective rate of interest is 26.8242%, and we want an equivalent interest rate compounded every day, say $k$ (in a 365-dayP year), then $(1 + k)^{365} = 1.268242$, so that $k = .00065126 = (1.268242)^{1/365} - 1$ (this is .065126% per day). The equivalent nominal annual interest rate compounded daily is denoted $i^{(365)}$, and $\frac{i^{(365)}}{365} = k$ is the daily compound interest rate, so that the nominal annual rate of interest compounded daily is $365 \times k = i^{(365)} = 365 \times .00065126 = .2377$ (23.77%).

Note that $i^{(365)} = 365 \times [(1.268242)^{1/365} - 1] = m \times [(1 + i)^{1/m} - 1]$, with $m = 365$. We can extend this to a compounding period of 6 hours ($m = 1460$, which means there are 1460 6-hour periods in one year with nominal rate $i^{(1460)}$), or 10 minutes ($m = 52,560$ 10-minute periods in a year, with $i^{(52,560)}$), one second ($m = 31,536,000$ seconds in one year, with $i^{(31,536,000)}$), etc. As the compounding time interval goes to 0 and the number of compounding periods per year, $m$, goes to $\infty$, $i^{(m)}$ will approach a limit $i^{(\infty)}$. The limit is referred to as **continuous compounding**, and the corresponding nominal annual rate limit $i^{(\infty)}$ is called the force **of interest**.

From an algebraic point of view, for a given annual effective interest rate $i$, as $m \to \infty$,
$i^{(m)} \to i^{(\infty)} = \lim_{m \to \infty} m \times [(1 + i)^{1/m} - 1] = ln(1 + i)$ (from basic calculus limits).
$i^{(\infty)}$ is called the **force of interest**, and is denoted $\delta$.

Also, $d^{(\infty)} = \lim_{m\to\infty} m \times [1-(1-d)^{1/m}] = -ln(1-d) = ln(1+i) = \delta$, so that the **force of discount is equal to the force of interest**. For equivalent rates and $n > m > 1$, we have the relationship $d < d^{(m)} < d^{(n)} < d^{(\infty)} = \delta = i^{(\infty)} < i^{(n)} < i^{(m)} < i$.
For example, for annual effective interest rate $i = .26842$, we have $\delta = ln(1.126842) = .2376$, and $d = .2115 < d^{(2)} = .2241 < d^{(12)} = .2353 < d^{(365)} = .237554 < \delta = .237632$

$$< i^{(365)} = .237709 < i^{(12)} = .24 < i^{(2)} = .2524 < i = .268242$$

The force of interest is a nominal annual interest (or discount) rate "convertible continuously" (or infinitely many times per year, $m = \infty$). Force of interest provides another equivalent way of describing investment growth. **For equivalent annual effective $i$, annual effective $d$ and force of interest (and discount) $\delta$, we have the following relationships:**

$\boldsymbol{\delta = ln(1+i)}$, $\boldsymbol{1+i = e^{\delta}}$, $\boldsymbol{\delta = -ln(1-d)}$ , $\boldsymbol{1-d = e^{-\delta} = v}$,
and $\boldsymbol{(1+i)^n = e^{n\delta}}$ and $\boldsymbol{v^n = e^{-n\delta} = (1-d)^n}$.

For annual effective rate of interest $i = .268242$, we have $\delta = ln(1.268242) = .237632$.
This is very close to $i^{(365)} = .237709$. Interest compounding every day is very close to being continuously compounded interest.

There is a more general way of defining the force of interest being earned by a fund at a particular point in time $t$. In general, suppose that an investment of amount $A(0) = 1$ is made at time $t = 0$, and $A(t)$ denotes the accumulated value at time $t > 0$. Suppose that we consider the interval of time from time $t$ to time $t+1$. Then the annual effective interest rate for that interval of time is $\frac{A(t+1)-A(t)}{A(t)} = \frac{\text{amount of interest earned from time } t \text{ to time } t+1}{\text{value of fund at time } t}$ .

In a similar way, the $\frac{1}{2}$-year interest rate for the period from time $t$ to time $t+\frac{1}{2}$ is $\frac{A(t+\frac{1}{2})-A(t)}{A(t)}$. We could describe this as a nominal annual rate of interest compounded twice per year; this would be of $2 \times \frac{A(t+\frac{1}{2})-A(t)}{A(t)} = \frac{A(t+\frac{1}{2})-A(t)}{\frac{1}{2}\times A(t)}$, since the nominal annual rate compounded 2 times per year is $2 \times \frac{1}{2}$-year effective rate. Note that this nominal annual rate of interest compounded twice per year is **based only on the actual behavior of the investment during the time period from time $t$ to time $t+\frac{1}{2}$** (and is not related to investment behavior from time $t+\frac{1}{2}$ to time 1).

This idea can be extended to the time period from time $t$ to time $t+\frac{1}{m}$. The interest rate for the $\frac{1}{m}$-th year period from time $t$ to time $t+\frac{1}{m}$ is $\frac{A(t+\frac{1}{m})-A(t)}{A(t)}$.

This can be described as a nominal annual rate of interest compounded $m$ times per year of $m \times \frac{A(t+\frac{1}{m})-A(t)}{A(t)} = \frac{A(t+\frac{1}{m})-A(t)}{\frac{1}{m}\times A(t)}$ (again this nominal rate is **based only on the behavior of the investment during the time period from time $t$ to time $t+\frac{1}{m}$**).

If we let $m$ get larger and larger, we are focusing on investment behavior in a smaller and smaller time interval from time $t$ to time $t+\frac{1}{m}$, but we are describing that investment behavior on a nominal annual basis. We can take the limit as $m \to \infty$, and the resulting limit is the **force of interest at time $t$**. The result is a little easier to formulate algebraically if we make the change of variable $h=\frac{1}{m}$, and take the limit as $h \to 0$.

The force of interest at time $t$ for this pattern of accumulation is defined to be

$$\boldsymbol{\delta_t = \frac{A'(t)}{A(t)} = \frac{d}{dt}\, ln[A(t)]}.$$

If accumulation is based on compound interest at annual effective rate $i$, then $A(t)=(1+i)^t$, and $\delta_t = \frac{A'(t)}{A(t)} = \frac{d}{dt}\, ln[A(t)] = \frac{d}{dt}\, ln(1+i)^t = \frac{d}{dt}\,[\,t \cdot ln(1+i)] = ln(1+i) = \delta$.
Thus, as seen above, compound interest at (constant) annual effective interest rate $i$ is equivalent to a constant force of interest of $\delta$, where $\delta = ln(1+i)$. If annual effective interest rates are $i_1$ and $i_2$ in years 1 and 2 then the equivalent forces of interest would be $\delta_1 = ln(1+i_1)$ and $\delta_2 = ln(1+i_2)$ in years 1 and 2. This applies to fractions of a year as well. Note that $\delta_t$ does not have to be constant for all $t$.

It is possible to express accumulated value and present value in terms of $\delta_t$ when the force of interest $\delta_t$ is specified. The accumulated value at time $n$ of an initial investment of 1 at time 0 is $\boldsymbol{e^{\int_0^n \delta_t\, dt}}$. This formulation follows form the fact (above) that $\delta_t = \frac{d}{dt}\, ln[A(t)]$. It follows from this that $\int_0^n \delta_t\, dt = \int_0^n \left(\frac{d}{dt}\, ln[A(t)]\right) dt = ln[A(n)] - ln[A(0)] = ln\left[\frac{A(n)}{A(0)}\right]$, and then $e^{\int_0^n \delta_t\, dt} = e^{ln\left[A(n)/A(0)\right]} = \frac{A(n)}{A(0)} = A(n)$ if $A(0)=1$. This is the accumulation factor for the period from time 0 to time $n$. The present value of 1 due at time $n$ is the inverse, which is $\boldsymbol{e^{-\int_0^n \delta_t\, dt}}$. These accumulation and present value formulations can be generalized as follows: if $n_1 < n_2$, the accumulated value at time $n_2$ of an investment of 1 made at time $n_1$ is $\boldsymbol{e^{\int_{n_1}^{n_2} \delta_t\, dt}}$, and the present value at time $n_1$ of amount 1 due at time $n_2$ is $\boldsymbol{e^{-\int_{n_1}^{n_2} \delta_t\, dt}}$.

The verbal interpretation of force of interest is as follows. $A'(t)$ is the instantaneous rate at time $t$ at which the total investment value is changing, and the ratio $\frac{A'(t)}{A(t)}$ is the instantaneous rate at which the investment is changing per dollar invested. It is also worth noting that if the force of interest has the form $\delta_t = \frac{f'(t)}{f(t)} = \frac{d}{dt}\, ln[f(t)]$, then the accumulated value at time $n$ of an amount of 1 invested at time 0 is $A(n) = \frac{f(n)}{f(0)}$. This has occasionally arisen on the FM exam.

If the force of interest is constant, $\delta_t = \delta$ for $0 \le t \le n$, the accumulation factor for the $n$-year period is $e^{n\delta}$ (and $\delta$ is equivalent to annual effective interest rate $i = e^{\delta} - 1$); the relationship between constant force of interest and equivalent nominal and annual effective rates of interest and discount is $e^{\delta/m} = (1+i)^{1/m} = 1 + \frac{i^{(m)}}{m}$ , and $e^{-\delta/m} = (1-d)^{1/m} = 1 - \frac{d^{(m)}}{m}$.

**Example 10:** On September 1, 2017, a bank is offering a 6-month investment with a nominal annual interest rate of 8% compounded semiannually. Smith invests \$10,000 into this investment. On March 1, 2018, Smith's investment matures and the bank offers another 6-month investment with a nominal annual interest rate of 12%. On that date, Smith invests the proceedings from the first investment into the new 6-month investment. Find Smith's equivalent annual effective return $j$ based on the first 6-month investment and then $k$, based on the second 6-month investment. Find Smith's annual effective return $i$ for the one year period of the combined investments. Find the equivalent force of interest $\delta_j$ on Smith's investment during the first 6-month period and then $\delta_k$ during the second 6-month period, then find Smith's equivalent force of interest $\delta$ for the one year period of the combined investments.

**Solution:** A nominal annual interest rate of 8% convertible semiannually is equivalent to an annual effective rate of $(1.04)^2 - 1 = .0816 = j$ and a nominal annual interest rate of 12% convertible semiannually is equivalent to an annual effective rate of $(1.06)^2 - 1 = .1236 = k$.
Over the full year, Smith's investment grows by a factor of $1.04 \times 1.06 = 1.1024 = 1 + i$.
The force of interest equivalent to the interest rate in the first 6-month investment is $\delta_j = ln(1.0816)$, since .0816 is the equivalent annual effective rate during the first 6 months.
Note that $\delta_j = ln(1.0816) = ln[(1.04)^2] = 2\, ln(1.04) = .078441$. During the second 6-month period, the equivalent force of interest it $\delta_k = ln(1.1236) = 2\, ln(1.06) = .116538$. The equivalent force of interest for the full year period is $\delta = ln(1.1024) = .097490$ . □

A point to note about the situation presented in Example 10 is that even though the 6-month investment starting on September 1, 2017 has a quoted nominal annual interest rate of 8% compounded semiannually, there is no guarantee that the same rate will be available after 6 months. The quotation on March 1, 2018 of the nominal annual rate compounded semiannually does not imply that it will continue beyond September 1, 2018.

Another minor point to note in Example 10 is that $i = .1024$ is not the simple average of .0816 and .1236 (the simple average is .1026, whereas $.1024$ is based on a "geometric average"). Note also that the force of interest for the full year is

$$\delta = ln(1.1024) = ln[1.04 \times 1.06] = ln(1.04) + ln(1.06)$$
$$= \frac{1}{2} \times [ln(1.0816) + ln(1.12360)] = \frac{1}{2} \times (\delta_j + \delta_k).$$

From this we see that the overall force of interest for the full year is the simple average of the force of interest during the first and second 6-month periods. This contrasts with the relationship between $i$ and $j$ and $k$. Keep in mind that the force of interest is a nominal rate of interest compounded continuously (infinitely often), but just as a nominal annual rate convertible semiannually can change from one six-month period to the next, as in Example 10, the force of interest $\delta_t$ can change over time.

If accumulation is based on simple interest at annual rate of interest $i$, then $A(t) = 1 + it$, and $\delta_t = \frac{A'(t)}{A(t)} = \frac{i}{1+it}$. **It is worth making a special note of this last comment**, as this form for force of interest that corresponds to simple interest has arisen occasionally on previous exam questions.

**Example 11 (SOA):** On 1/1/2017, Kelly deposits $X$ into a bank account. The account is credited with simple interest at the rate of 10% per year. On the same date, Tara deposits $X$ into a different bank account. The account is credited with interest using a force of interest $\delta_t = \frac{2t}{t^2+k}$. From the end of the 4th year until the end of the 8th year, both accounts earn the same dollar amount of interest. Calculate $k$.

**Solution:** When working with simple interest, it is important to identify the time point $t = 0$. Then, for amount $X$ invested at time 0, the accumulated value at time $t > 0$ is $X(1 + it)$ ($t$ measured in years). In this case, $t = 0$ corresponds to 1/1/2017.

Since there were no additional deposits beyond the first deposit in either account, the change in balance from one time point to another is due only to interest earned on each account.

The dollar amount of interest earned on Kelly's account from the end of the 4th year ($t = 4$) to the end of the 8th year ($t = 8$) is the difference in the bank balance between those two dates, which is $X \times [A_K(8) - A_K(4)]$ ($K$ for Kelly). This is
$X \times [1 + (.1)(8)] - X \times [1 + (.1)(4)] = .4X$.
The dollar amount of interest earned in Tara's account will be $X \times [A_T(8) - A_T(4)]$ ($T$ for Tara). Given the force of interest for Tara's account, we have $S_T(n) = e^{\int_0^n \frac{2t}{t^2+k}\,dt} = \frac{n^2+k}{k}$.
The amount of interest earned from the end of the 4th to the end of the 8th year is
$X \times [\frac{8^2+k}{k} - \frac{4^2+k}{k}] = \frac{48X}{k}$. Setting this equal to Kelly's interest results in $.4X = \frac{48X}{k}$,
so that $k = 120$. □

An important point to note about Example 11 is the following. When we calculate the balance in Kelly's account at the end of year 4 we get $A_K(4) = X \times [1 + (.1)(4)]$. When we calculate the balance in that account at the end of year 8 we get $A_K(8) = X \times [1 + (.1)(8)]$. If we look at the balance at time 4 as compared to the balance at time 8, it is not the case that the account is earning simple interest from time 4 to time 8. If we had simple interest starting at time 4 applied to the balance at time 4, then 4 years later the balance would be
$A_K(4) \times [1 + (.1)(4)] = X \times [1 + (.1)(4)] \times [1 + (.1)(4)]$.
But this is not equal to the actual balance at time 8, which is $X \times [1 + (.1)(8)]$.
Simple interest is tied to the point at which simple interest begins, so that must be specified in a situation involving simple interest. This is illustrated again in the following example.

**Example 12 (SOA):** At a force of interest $\delta_t = \frac{.1}{1+.1t}$, $0 \le t \le 14$, the following payments have the same present value:
(i) $X$ at the end of year 5 plus $2X$ at the end of year 10; and
(ii) $Y$ at the end of year 14.
Calculate $\frac{Y}{X}$.
**Solution:** The given force of interest corresponds to simple interest accumulation at annual rate $i = .1$, and we can use simple interest for present value, but it is important that all simple interest accumulation is from time $t = 0$. The total present value at time 0 of the payments in (i) is $X(\frac{1}{1+5(.1)}) + 2X(\frac{1}{1+10(.1)}) = \frac{5}{3}X$, and the present value at time 0 of the payment in (ii) is $Y(\frac{1}{1+14(.1)}) = \frac{5}{12}Y$. Setting these equal results in $\frac{Y}{X} = 4$. □